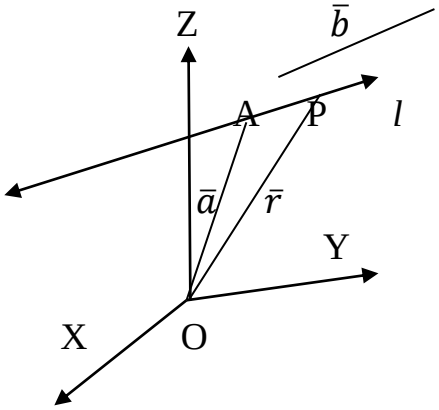
LINE

Vector equation of a line:

Equation of a line through a given point and parallel to the given vector:

Let $\bar{a}$ be the position vector of the given point A.
Let l be the line which passes through the point A and is parallel to a given vector $\bar{b}$.
Let $\bar{r}$ be the position vector of any point P on the line l
Then $\overline{AP}$ is parallel to the vector $\bar{b}$
i.e. $\overline{AP} = \lambda\bar{b}$, where λ is a scalar.
But $\overline{AP} = \overline{OP} - \overline{OA}$
$\therefore\ \lambda\bar{b} = \bar{r} - \bar{a}$



Hence, the vector equation of the line is given by

$$\bar{r} = \bar{a} + \lambda\,\bar{b} \qquad \text{.....(1)}$$

This equation is referred as parametric form of the line.

Remark:

(i) If $\bar{b} = a_1\hat{\imath} + b_1\hat{\jmath} + c_1\hat{k}$, then a_1, b_1, c_1 are direction ratios of the line and conversely, if a_1, b_1, c_1 are direction ratios of a line, then $\bar{b} = a_1\hat{\imath} + b_1\hat{\jmath} + c_1\hat{k}$ will be parallel to the line.

(ii) Cartesian form of an equation of a line. Let A $\equiv (x_1, y_1, z_1)$ be a point on the line and a, b, c be direction ratios of the line. Consider any point P(x, y, z) on the line.

Then $\bar{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

$\bar{a} = x_1\hat{\imath} + y_1\hat{\jmath} + z_1\hat{k}$

and $\bar{b} = a_1\hat{\imath} + b_1\hat{\jmath} + c_1\hat{k}$

substituting these values in equation (1) and equating the coefficients of $\hat{\imath}, \hat{\jmath}, \hat{k}$, we get

$$x = x_1 + \lambda a_1, \quad y = y_1 + \lambda b_1, \quad z = z_1 + \lambda c_1 \qquad \text{..... (2)}$$

These are the parametric equations of the line. Eliminating the parameter λ from (2), we get

$$\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1} \qquad \text{.....(3)}$$

These are the Cartesian equations of the line.

(iii) If $l, m. n$ are direction cosines of the line, the equations of the line are

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} \qquad \text{.....(4)}$$

(iv) Vector equations of a line passing through the origin and parallel to the vector $\bar{b}$ is

$$\bar{r} = \lambda\,\bar{b} \qquad (\text{since } \bar{a} = 0)$$

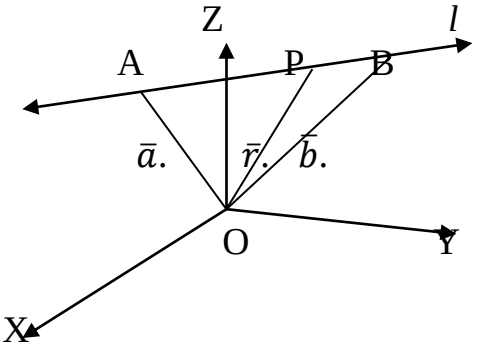
(v) Vector equation of a line passing through the point A$(\bar{a})$ and parallel to the given vector $\bar{b}$ can also be expressed as

$(\bar{r} - \bar{a}) \times \bar{b} = 0.$

(vi) The coordinates of any point on the line $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ are $(x_1 + \lambda a_1,\ y_1 + \lambda b_1,\ z_1 + \lambda c_1)$, where $\lambda \epsilon R$.

Vector equation of the line passing through two gives points:

Let A(x_1, y_1, z_1) and B(x_2, y_2, z_2) be any two points on line l.



Let $\bar{a}$ and $\bar{b}$ be the position vectors of two points A and B respectively.

Let $\bar{r}$ be the position vector of any point P (x, y, z) on the line l.

Then $\overline{AP} = \bar{r} - \bar{a}$ and

$\overline{AB} = \bar{b} - \bar{a}$.

The vectors $\overline{AP}$ and $\overline{AB}$ are collinear

Therefore

$\overline{AP} = \lambda\,(\overline{AB})$

$\therefore \quad \bar{r} - \bar{a} = \lambda\,(\bar{b} - \bar{a})$

$\therefore \quad \bar{r} = \bar{a} + \lambda\,(\bar{b}\text{-}\bar{a})$

i.e. $\bar{r} = (1-\lambda)\bar{a} + \lambda\,\bar{b}$ } ……(1)

This is the vector equation of the line passing through the points A($\bar{a}$) and B($\bar{b}$).

Remark:

(i) If $\bar{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$, $\bar{a} = x_1\hat{\imath} + y_1\hat{\jmath} + z_1\hat{k}$ and $\bar{b} = x_2\hat{\imath} + y_2\hat{\jmath} + z_2\hat{k}$ from (1), we get

$$x\hat{\imath} + y\hat{\jmath} + z\hat{k} = x_1\hat{\imath} + y_1\hat{\jmath} + z_1\hat{k} + \lambda\left[(x_2 - x_1)\hat{\imath} + (y_2 - y_1)\hat{\jmath} + (z_2 - z_1)\hat{k}\right]$$

Equating the coefficients of $\hat{\imath}, \hat{\jmath}, \hat{k}$ we get,

$$x = x_1 + \lambda(x_2 - x_1), y = y_1 + \lambda(y_2 - y_1), z = z_1 + \lambda(z_2 - z_1),$$

on eliminating λ, we obtain

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

Which are the equations of the line in Cartesian form passing through two points A (x_1, y_1, z_1) and B(x_2, y_2, z_2).

(ii) The equation $\bar{r} = (1-\lambda)\bar{a} + \lambda\,\bar{b}$ is called the parametric vector equation of a line passing through two points, where λ is parameter.

(iii) The vector equation of a line passing through two points A($\bar{a}$) and B($\bar{b}$) is also given by $(\bar{r} - \bar{a}) \times (\bar{b} - \bar{a}) = 0$. This equation is also referred as non-parametric equation of the line.

(iv) If $\bar{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

$\bar{a} = a_1\hat{\imath} + b_1\hat{\jmath} + c_1\hat{k}$ and $\bar{b} = a_2\hat{\imath} + b_2\hat{\jmath} + c_2\hat{k}$

Then $\bar{r} = (1-\lambda)\bar{a} + \lambda\,\bar{b}$ gives $\frac{x-a_1}{a_2-a_1} = \frac{y-b_1}{b_2-b_1} = \frac{z-c_1}{c_2-c_1} = \lambda$

This is the symmetric form of equation of a line.

Examples

1) Find the vector equation of the line through A(3, 4, -7) and B (6, -1, 1). Also find the Cartesian form.

Sol: The line pass through the pt. A(3, 4, -7) and B(6, -1, 1)

Let $\bar{a}$ and $\bar{b}$ be the position vector of the points A and B.

$\bar{a} = 3\hat{\imath} + 4\hat{\jmath} - 7\hat{k}$ and $\bar{b} = 6\hat{\imath} - \hat{\jmath} + \hat{k}$

$\bar{b} - \bar{a} = (6\hat{\imath} - \hat{\jmath} + \hat{k}) - (3\hat{\imath} + 4\hat{\jmath} - 7\hat{k})$

$\bar{b} - \bar{a} = 3\hat{\imath} - 5\hat{\jmath} + 8\hat{k}$

The vector equation of the line passing through the points A($\bar{a}$) and B($\bar{b}$) is

$\bar{r} = \bar{a} + \lambda(\bar{b} - \bar{a})$

$\bar{r} = (3\hat{\imath} + 4\hat{\jmath} - 7\hat{k}) + \lambda\,(3\hat{\imath} - 5\hat{\jmath} + 8\hat{k})$

The Cartesian form of the equation of the line passing through (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

$$\therefore \frac{x-3}{6-3} = \frac{y-4}{-1-4} = \frac{z-(-7)}{1-(-7)}$$

i.e. $\frac{x-3}{3} = \frac{x-4}{-5} = \frac{z+7}{8}$

2) Find the equation of the line in vector form passing through the point (4, -2, 5) and parallel to the vector $3\hat{\imath} - \hat{\jmath} + 2\hat{k}$. Hence, find the equation in Cartesian form.

Sol: Given that the line pass through the point A(4, -2, 5) and parallel to the vector $\bar{b} = 3\hat{\imath} - \hat{\jmath} + 2\hat{k}$.

Let a be the position vector of the point A

$\therefore \bar{a} = 4\hat{\imath} - 2\hat{\jmath} + 5\hat{k}$

$\therefore$ The vector equation of line passing through the pt. A($\bar{a}$) and parallel to vector $\bar{b}$ is

$\bar{r} = \bar{a} + \lambda\bar{b}$ where λ is scalar

$\bar{r} = (4\hat{\imath} - 2\hat{\jmath} + 5\hat{k}) + \lambda\,(3\hat{\imath} - \hat{\jmath} + 2\hat{k}.)$

If $\bar{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ then

$x\hat{\imath} + y\hat{\jmath} + z\hat{k} = (4\hat{\imath} - 2\hat{\jmath} + 5\hat{k}) + \lambda\,(3\hat{\imath} - \hat{\jmath} + 2\hat{k})$

$$(x-4)\hat{\imath} + (y+2)\hat{\jmath} + (z-5)\hat{k} = 3\lambda\hat{\imath} - \lambda\hat{\jmath} + 2\lambda\hat{k}$$

Comparing the coefficients of $\hat{\imath}, \hat{\jmath}, \hat{k}$ on both sides.

$x - 4 = 3\lambda, y + 2 = -\lambda, z - 5 = 2\lambda$

$\frac{x-4}{3} = \frac{y+2}{-1} = \frac{z-5}{2} = \lambda$

∴ Cartesian form of the equation of line is $\frac{x-4}{3} = \frac{y+2}{-1} = \frac{z-5}{2}$

3) Find the equation of line passing through the point (5, 4, 3) and having direction ratios -3, 4, 2.

Sol: Given that the line pass through the point A(5, 4, 3) and having direction ratios -3, 4, 2

Let $\bar{a}$ be the position vector of pt. A

$\bar{a} = 5\hat{\imath} + 4\hat{\jmath} + 3\hat{k}$

Let $\bar{b}$ be the vector parallel to the line whose direction ratios are -3, 4, 2

$\bar{b} = -3\hat{\imath} + 4\hat{\jmath} + 2\hat{k}$

∴ The vector equation of the line passing through A($\bar{a}$) and parallel to $\bar{b}$ is

$\bar{r} = \bar{a} + \lambda\bar{b}$

$\bar{r} = (5\hat{\imath} + 4\hat{\jmath} + 3\hat{k}) + \lambda\,(-3\hat{\imath} + 4\hat{\jmath} + 2\hat{k})$

4) The Cartesian equation of line is $\frac{x-6}{2} = \frac{y+4}{7} = \frac{z-5}{3}$ find the vector equation of the line. (March 14)

Sol: The Cartesian equation of line is

$\frac{x-6}{2} = \frac{y+4}{7} = \frac{z-5}{3}$

∴ The line is passing through the point A(6, -4, 5) and having direction ratios 2, 7, 3

Let $\bar{a}$ be the position vector of the point A and $\bar{b}$ be the vector parallel to the line.

∴ $\bar{a} = 6\hat{\imath} - 4\hat{\jmath} + 5\hat{k},\ \bar{b} = 2\hat{\imath} + 7\hat{\jmath} + 3\hat{k}$

The vector equation of line is

$\bar{r} = \bar{a} + \lambda\bar{b}$

$\bar{r} = (6\hat{\imath} - 4\hat{\jmath} + 5\hat{k}) + \lambda\,(2\hat{\imath} + 7\hat{\jmath} + 3\hat{k})$

5) Find the vector equation of a line passing through the point $(\hat{\imath} + 2\hat{\jmath} + 3\hat{k})$ and perpendicular to the vector $(\hat{\imath} + \hat{\jmath} + \hat{k})$ and $(2\hat{\imath} - \hat{\jmath} + \hat{k})$.

Sol: Let $\bar{a} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k},\ \bar{b} = \hat{\imath} + \hat{\jmath} + \hat{k},\ \bar{c} = 2\hat{\imath} - \hat{\jmath} + \hat{k}$

The vector perpendicular to the vectors $\bar{b}$ and $\bar{c}$ is given by

$$\bar{b} \times \bar{c} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 1 & 1 \\ 2 & -1 & 1 \end{vmatrix} = \hat{\imath}(1+1) - \hat{\jmath}(1-2) + \hat{k}(-1-2) = 2\hat{\imath} + \hat{\jmath} - 3\hat{k}$$

Since line is perpendicular to the vector $\bar{b}$ and $\bar{c}$ it is parallel to $\bar{b} \times \bar{c}$.

The vector equation of the line passing through A($\bar{a}$) and parallel to $\bar{b} \times \bar{c}$ is

$\bar{r} = \bar{a} + \lambda\,(\bar{b} \times \bar{c})$

$\therefore$ Vector equation of the required line is

$\bar{r} = (\hat{\imath} + 2\hat{\jmath} + 3\hat{k}) + \lambda\,(2\hat{\imath} + \hat{\jmath} - 3\hat{k})$.

6) Find the vector equation of the line passing through the point (-1, -1, 2) and parallel to the line $2x - 2 = 3y + 1 = 6z - 2$

Sol: Let $\bar{a}$ be the position vector of the point A(-1, -1, 2)

$\therefore \bar{a} = -\hat{\imath} - \hat{\jmath} + 2\hat{k}$

Given equation of line is

$$2x - 2 = 3y + 1 = 6z - 2$$

$$2(x - 1) = 3\left(y + \frac{1}{3}\right) = 6(z - \frac{1}{3})$$

$$\frac{x-1}{\frac{1}{2}} = \frac{y+\frac{1}{3}}{\frac{1}{3}} = \frac{z-\frac{1}{3}}{\frac{1}{6}}$$

$\therefore$ Direction ratios are

$\frac{1}{2}, \frac{1}{3}, \frac{1}{6}$ i.e. 3, 2, 1

Let $\bar{b}$ be the vector parallel to required line

$\bar{b} = 3\hat{\imath} + 2\hat{\jmath} + \hat{k}$

$\therefore$ The vector equation of the line passing through A($\bar{a}$) and parallel to $\bar{b}$ is

$\bar{r} = \bar{a} + \lambda\bar{b}$

$\bar{r} = (-\hat{\imath} - \hat{\jmath} + 2\hat{k}) + \lambda\,(3\hat{\imath} + 2\hat{\jmath} + \hat{k})$

7) The Cartesian equation of a line is $3x - 1 = 6y + 2 = 1 - z$. Find the direction ratios and write its equation in vector form.

Sol: The Cartesian equation of a line is

$3x - 1 = 6y + 2 = 1 - z$

$3\left(x - \frac{1}{3}\right) = 6\left(y + \frac{1}{3}\right) = -(z - 1)$

$\frac{x - 1/3}{1/3} = \frac{y+1/3}{1/6} = \frac{z - 1}{-1}$

$\therefore$ The line pass through the point A($\frac{1}{3}, -\frac{1}{3}$, 1) and having

Direction ratios $\frac{1}{3}, \frac{1}{6}, -1$ i.e. 2, 1, -6

Let $\bar{a}$ be the position vector of the point A and $\bar{b}$ be the vector parallel to the line.

$\bar{a} = \frac{1}{3}\hat{\imath} - \frac{1}{3}\hat{\jmath} + \hat{k}$ and $\bar{b} = 2\hat{\imath} + \hat{\jmath} - 6\hat{k}$

$\therefore$ vector equation is

$\bar{r} = \bar{a} + \lambda\bar{b}$

$r = (\frac{1}{3}\hat{\imath} - \frac{1}{3}\hat{\jmath} + \hat{k}) + \lambda\,(2\hat{\imath} + \hat{\jmath} - 6\hat{k})$

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8) Show that the points whose position vectors are $5\hat{\imath} + 5\hat{k}$, $-4\hat{\imath} + 3\hat{\jmath} - \hat{k}$ and $2\hat{\imath} + \hat{\jmath} + 3\hat{k}$ are collinear.

Sol: Let A, B, C be the points with position vector $\bar{a}$, $\bar{b}$, $\bar{c}$

$\bar{a} = 5\hat{\imath} + 5\hat{k}$, $\bar{b} = -4\hat{\imath} + 3\hat{\jmath} - \hat{k}$, $\bar{c} = 2\hat{\imath} + \hat{\jmath} + 3\hat{k}$

vector equation of line AB is

$\bar{r} = \bar{a} + \lambda(\bar{b} - \bar{a})$

$\bar{b} - \bar{a} = -4\hat{\imath} + 3\hat{\jmath} - \hat{k}) - (5\hat{\imath} + 5\hat{k}) = -9\hat{\imath} + 3\hat{\jmath} - 6\hat{k}$

$\therefore$ $\bar{r} = (5\hat{\imath} + 5\hat{k}) + \lambda\,(-9\hat{\imath} + 3\hat{\jmath} - 6\hat{k})$ …..(1)

Now the points A, B, C are collinear if the point C lies on the line AB i.e. the position vector $\bar{c}$ of C must satisfy the equation (i) for some scalar λ.

$\therefore$ Replacing $\bar{r}$ by $\bar{c}$ in equation (1) we get

$2\hat{\imath} + \hat{\jmath} + 3\hat{k} = (5\hat{\imath} + 5\hat{k}) + \lambda\,(-9\hat{\imath} + 3\hat{\jmath} - 6\hat{k})$

$2\hat{\imath} + \hat{\jmath} + 3\hat{k} - (5\hat{\imath} + 5\hat{k}) = \lambda\,(-9\hat{\imath} + 3\hat{\jmath} - 6\hat{k})$

$-3\hat{\imath} + \hat{\jmath} - 2\hat{k} = \lambda\,(-9\hat{\imath} + 3\hat{\jmath} - 6\hat{k})$

$\frac{1}{3}(-9\hat{\imath} + 3\hat{\jmath} - 6\hat{k}) = \lambda\,(-9\hat{\imath} + 3\hat{\jmath} - 6\hat{k})$

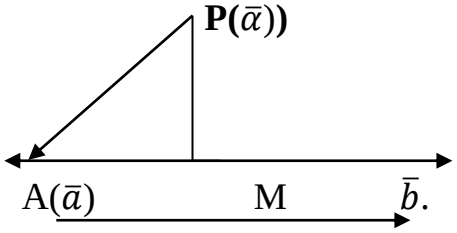
$\therefore$ $\lambda = \frac{1}{3}$

Thus the points A,B, C are collinear.

Examples for practice

1) Find the vector equation of a line which passes through the point with position vector $4\hat{\imath} - \hat{\jmath} + 2\hat{k}$ and is in the direction of $-2\hat{\imath} + \hat{\jmath} + \hat{k}$. Also reduce the equation to Cartesian form.
2) Find the vector equation of the line passing through the point with position vector $2\hat{\imath} + \hat{\jmath} - \hat{k}$ and parallel to the line joining the points $-\!-\hat{\imath} + \hat{\jmath} + 4\hat{k}$ and $\hat{\imath} + 2\hat{\jmath} + 2\hat{k}$. Also find the Cartesian form of this equation.
3) Find the Cartesian equation of line passing through the points A(4, 2, 1) and B (2, -1, 3). Also, reduce it to vector form.
4) Find the angle between the lines $\frac{x-1}{4} = \frac{y-3}{1} = \frac{z}{8}$ and $\frac{x-2}{2} = \frac{y+1}{2} = \frac{z-4}{1}$.

Distance of a point from a line:



Let P ($\bar{a}$) be a point and

Let $\bar{r} = \bar{a} + \lambda\bar{b}$ be the vector equation of a line.

The line passes through a point A($\bar{a}$).

Let M be the foot of the perpendicular drawn from the point P ($\bar{a}$) on the line $\bar{r} = \bar{a} + \lambda\bar{b}$.

Now $\overline{AM} \parallel \bar{b}$

$\therefore \overline{AM} = \lambda\bar{b}$

and $l(\text{AM}) = \frac{\overline{AP}.\overline{AM}}{|\overline{AM}|} = \frac{(\bar{\alpha}-\bar{a}).\overline{AM}}{|\overline{AM}|} = \frac{(\bar{\alpha}-\bar{a}).\bar{b}}{|\bar{b}|}$ ……..(1)

Now, ΔAMP is right angled triangle.

$\therefore \quad (\overline{PM})^2 = (\overline{AP})^2 - (\overline{AM})^2 = |\bar{\alpha} - \bar{a}|^2 - [\frac{(\bar{\alpha}-\bar{a}).\bar{b}}{|\bar{b}|}]^2$

$\therefore \quad PM = \sqrt{|\bar{\alpha} - \bar{a}|^2 - [\frac{(\bar{\alpha}-\bar{a}).\bar{b}}{|\bar{b}|}]^2}$

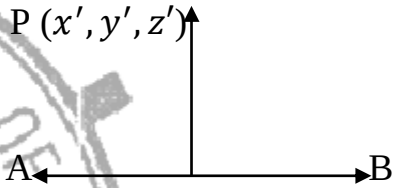
Hence, distance of a point P (α) from a line is $\bar{r} = \bar{a} + \lambda\bar{b}$.

is $\sqrt{|\bar{\alpha} - \bar{a}|^2 - [\frac{(\bar{\alpha}-\bar{a}).\bar{b}}{|\bar{b}|}]^2}$

Cartesian form:

Let P (x', y', z') be a given point and

Let $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$ be given line.



Let M be the foot of the perpendicular drawn from P(x', y', z')

On the line $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$

Let coordinates of M be $(x_1 + \lambda a,\ y_1 + \lambda b,\ z_1 + \lambda c)$ then direction ratios of PM are

$(x_1 + \lambda a - x',\ y_1 + \lambda b - y',\ z_1 + \lambda c - z')$

The direction ratios of AB are a,b,c.

PM is perpendicular to AB

$\therefore \lambda = \frac{a(x'-x_1)+b(y'-y_1)+c(z'-z_1)}{a^2+b^2+c^2}$

Putting this value of λ in $(x_1 + \lambda a,\ y_1 + \lambda b,\ z_1 + \lambda c)$ we obtain coordinate of M, and using distance formula we get the length PM.

Skew Lines:

If two lines in space interest at a point, then the shortest distance between them is zero. Also, if two lines in space are parallel, then the shortest distance between them is the perpendicular distance between them.

Further, in space, there are lines which are neither intersecting nor parallel. In fact, such pair of lines are called non-coplanar or skew lines.

Distance between skew lines:

l_1 and l_2 are two skew lines therefore there is only one line perpendicular to each of lines l_1 and l_2 which is known as the line of shortest distance. Here distance PQ is called to be shortest distance.

Let l_1 and l_2 be two lines whose equations are

$\bar{r} = \bar{a}_1 + \lambda\bar{b}_1$ and $\bar{r} = \bar{a}_2 + \lambda\bar{b}_2$ respectively.

$\therefore l_1$ passes through A($\bar{a}_1$) and l_2 passes through B ($\bar{a}_2$)).

Also, l_1 and l_2 are parallel to $\bar{b}_1$ and $\bar{b}_1$ respectively.

$\overline{PQ}$ is perpendicular to both l_1 and l_2

$\therefore \overline{PQ}$ is parallel to $\bar{b}_1 \times \bar{b}_2$.

Let $\hat{n}$ be a unit vector along $\overline{PQ}$, then $\hat{n} = \pm \frac{\bar{b}_1 \times \bar{b}_2}{|\bar{b}_1 \times \bar{b}_2|}$

$\therefore \overline{PQ}$ = Projection of $\overline{AB}$ on PQ

$\therefore \overline{PQ} = \overline{AB}.\hat{n}$

$= (\bar{a}_2 - \bar{a}_1).\frac{\bar{b}_1 \times \bar{b}_2}{|\bar{b}_1 \times \bar{b}_2|} = \frac{(\bar{a}_2 - \bar{a}_1).(b_1 \times \bar{b}_2)}{|\bar{b}_1 \times \bar{b}_2|}$

Hence, distance PQ $= \left|\frac{(\bar{a}_2 - \bar{a}_1).(b_1 \times \bar{b}_2)}{|\bar{b}_1 \times \bar{b}_2|}\right|$

Cartesian form: The shortest distance between two skew lines

$\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$ is

$$\left| \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2c_1)^2}} \right|$$

Distance between parallel lines:

If two lines l_1 and l_2 are parallel,

then they are coplanar.

Let the lines be given by

$\bar{r} = \bar{a}_1 + \lambda\bar{b}$ and ……(1)

$\bar{r} = \bar{a}_2 + \mu\bar{b}$ ……..(2)

where $\bar{a}_1$ is the position vector of a point A on l_1 and $\bar{a}_2$ be the position vector of a point B on l_2

As l_1, l_2 are coplanar, if the foot of the perpendicular from B on the line l_1 is M, then the distance between the lines l_1 and l_2 is |BM|.

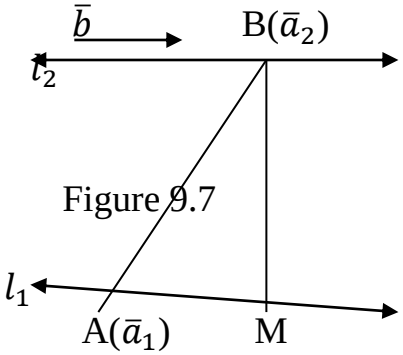


Figure 9.7

Let θ be the angle between The vectors $\overline{AB}$ and $\bar{b}$

Thus, $\bar{b} \times \overline{AB} = (|\bar{b}|\,|\overline{AB}| \sin\theta)\hat{n}$ ……(3)

Where $\hat{n}$ is unit vector perpendicular to
the plane of $\bar{b}$ and $\overline{AB}$

From triangle AMB, $\sin\theta = \frac{BM}{AB} = \frac{BM}{|\overline{AB}|} \therefore |\overline{AB}| \sin\theta = \text{BM}$

Now, $\overline{AB} = \bar{a}_2 - \bar{a}_1$

Therefore, from (3), we get

$$\bar{b} \times (\bar{a}_2 - \bar{a}_1) = |\bar{b}|\ \text{BM}\ \hat{n}$$

i.e. $$|\bar{b} \text{ x } (\bar{a}_2 - \bar{a}_1)| = |\bar{b}|\ \text{BM} \qquad (|\hat{n}|=1)$$

Hence, the distance between the parallel lines is d where

$$\text{d} = \text{BM} = \left|\frac{\bar{b} \times (\bar{a}_2 - \bar{a}_1)}{|\bar{b}|}\right| = |(\bar{a}_2 - \bar{a}_1) \times \hat{b}|$$

Remark: The lines are intersecting, if shortest distance is zero.

i.e. $$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

Examples

1) Find the length of the perpendicular from $(2, -3, 1)$ to the line $\frac{x+1}{2} = \frac{y-3}{3} = \frac{z+2}{-1}$.

Sol: Let M be the foot of perpendicular drawn from the point P$(2, -3, 1)$ to the given line

The co-ordinate of any point on the line $\frac{x+1}{2} = \frac{y-3}{3} = \frac{z+2}{-1}$ are $2\lambda - 1, 3\lambda + 3, -\lambda - 2$

Let M $\equiv (2\lambda - 1, 3\lambda + 3, -\lambda - 2)$ …… (i)

Direction ratios PM are

$2\lambda - 1 - 2, \quad 3\lambda + 3 + 3, \quad -\lambda - 2 - 1$

i.e. $2\lambda - 3, \ 3\lambda + 6, \quad -\lambda - 3$

Direction ratios of the given line are $2, 3, -1$.

Since PM is perpendicular to the given line.

$2(2\lambda - 3) + 3(3\lambda + 6) - 1(-\lambda - 3) = 0$

$4\lambda - 6 + 9\lambda + 18 + \lambda + 3 = 0$

$14\lambda + 15 = 0$

$\lambda = \frac{-15}{14}$

Putting $\lambda = \frac{-15}{14}$ in (i) we get

M $\equiv \left(\frac{-44}{14}, \frac{-3}{14}, \frac{-13}{14}\right)$

$$\therefore \text{PM} = \sqrt{\left(\frac{-44}{14} - 2\right)^2 + \left(\frac{-3}{14} + 3\right)^2 + \left(\frac{-13}{14} - 1\right)^2}$$

$$= \sqrt{\left(\frac{-72}{14}\right)^2 + \left(\frac{39}{14}\right)^2 + \left(\frac{-27}{14}\right)^2}$$

$$= \frac{\sqrt{5184+1521+729}}{14}$$

$$\text{PM} = \frac{\sqrt{7334}}{14} \text{ units}$$

2) Find the shortest distance between the lines
$\bar{r} = (4\hat{\imath} - \hat{\jmath}) + \lambda\,(\hat{\imath} + 2\hat{\jmath} - 3\hat{k}\ )$ and $\bar{r} = (\hat{\imath} - \hat{\jmath} + 2\hat{k})\ ) + \mu\,(\hat{\imath} + 4\hat{\jmath} - 5\hat{k})$.

Sol: We know that the shortest distance between the lines

$\overline{r} = \bar{a}_1 + \lambda\bar{b}_1$ and ……(1)

$\bar{r} = \bar{a}_2 + \lambda\overline{b_2}$ is

$$d = \left|\frac{(\bar{b}_1 \times \bar{b}_2).(\bar{a}_2 - \bar{a}_1)}{|\bar{b}_1 \times \bar{b}_2|}\right|$$

Here $\overline{a_1} = 4\hat{\imath} - \hat{\jmath}\,, \overline{a_2} = \hat{\imath} - \hat{\jmath} + 2\hat{k}$

$\bar{b_1} = \hat{\imath} + 2\hat{\jmath} - 3\hat{k}\ , \overline{b_2} = \hat{\imath} + 4\hat{\jmath} - 5\hat{k}$

$$\bar{b}_1 \times \bar{b}_2 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & -3 \\ 1 & 4 & -5 \end{vmatrix} = \hat{\imath}(-10+12) - \hat{\jmath}\,(-5+3)\ + \hat{k}\,(4\ -2)$$

$$= 2\hat{\imath} + 2\hat{\jmath} + 2\hat{k}$$

$$\bar{a}_2 - \bar{a}_1 = (\hat{\imath} - \hat{\jmath} + 2\hat{k}) - (4\hat{\imath} - \hat{\jmath}) = -3\hat{\imath} + 2\hat{k}$$

$$(\bar{b}_1 \times \bar{b}_2).(\bar{a}_2 - \bar{a}_1) = (2\hat{\imath} + 2\hat{\jmath} + 2\hat{k}).(-3\hat{\imath} + 2\hat{k})$$
$$= 2(-3)\ +\ 2(0)\ +\ 2(2)$$
$$= -6 + 4$$
$$= -2$$

$$\left|\bar{b}_1 \times \bar{b}_2\right| = \sqrt{(2)^2 + (2)^2 + (2)^2}$$
$$= \sqrt{4+4+4} = 2\sqrt{3}$$

∴ shortest distance $= \left|\frac{-2}{2\sqrt{3}}\right| = \frac{1}{\sqrt{3}}$ units.

3) Find the shortest distance between the lines
$\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$ and $\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$.(Oct.13)

Sol: shortest distance between the lines
$\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$ is

$$\left|\frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}}\right|$$

Given equations are $\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$ and $\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$

$\therefore\ x_1 = -1,\quad y_1 = -1, z_1 = -1, x_2 = 3, y_2 = 5, z_2 = 7, a_1 = 7, b_1 = -6, c_1 = 1, a_2 = 1, b_2 = -2, c_2 = 1$

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \begin{vmatrix} 4 & 6 & 8 \\ 7 & -6 & 1 \\ 1 & -2 & 1 \end{vmatrix}$$

$$= 4(-6+2) - 6(7-1) + 8(-14+6)$$

$$= -16 - 36 - 64 = -116$$

$$(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2$$

$$= (-6+2)^2 + (1-7)^2 + (-14+6)^2 = 16 + 36 + 64 = 116$$

Shortest distance $= \left|\frac{-116}{\sqrt{116}}\right| = \sqrt{116}$ units.

4) If the lines $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-1}{4}$ and $\frac{x-3}{1} = \frac{y-k}{2} = \frac{z}{1}$ interest then find the value of k.

Sol: The lines $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$ intersect if

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

Given equations are $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-1}{4}$ and $\frac{x-3}{1} = \frac{y-k}{2} = \frac{z}{1}$

$\therefore\ x_1 = 1,\quad y_1 = -1, z_1 = 1, x_2 = 3, y_2 = k, z_2 = 0, a_1 = 2, b_1 = 3, c_1 = 4, a_2 = 1, b_2 = 2, c_2 = 1$

Since the line intersect

$$\begin{vmatrix} 2 & k+1 & -1 \\ 2 & 3 & 4 \\ 1 & 2 & 1 \end{vmatrix} = 0$$

$$2(3-8) - (k+1)(2-4) - 1(4-3) = 0$$

$$-10 - (k+1)(-2) - 1 = 0$$

$$2k - 9 = 0$$

$$k = \frac{9}{2}$$